Warwick Mathematics Exchange

MA244/MA258

Analysis III

2023, April 21st

Desync, aka The Big Ree

Contents

Introduction

Analysis is the study of limits and related concepts - notably, sequences, series, differentiation and integration. In *Analysis III*, we formalise the foundations of integration using Darboux sums, before exploring sequences and series of functions. We end with contour integrals and complex analysis, which excluded from MA258.

This document is intended to broadly cover all the topics within the (Mathematical) Analysis III modules. All knowledge and algorithms contained within the module guide for MA131 will be assumed as prior knowledge in this document.

Disclaimer: I make *absolutely no guarantee* that this document is complete nor without error. In particular, any content covered exclusively in lectures (if any) will not be recorded here. This document was written during the 2022 academic year, so any changes in the course since then may not be accurately reflected.

Notes on formatting

New terminology will be introduced in *italics* when used for the first time. Named theorems will also be introduced in *italics*. Important points will be **bold**. Common mistakes will be underlined. The latter two classifications are under my interpretation. YMMV.

Content not taught in the course will be outlined in the margins like this. Anything outlined like this is not examinable, but has been included as it may be helpful to know alternative methods to solve problems.

The table of contents above, and any inline references are all hyperlinked for your convenience.

Scalars are written in lowercase italics, c, or using greek letters.

Vectors are written in lowercase bold, $\mathbf{v}$, or rarely overlined, $\overleftrightarrow{v}$, where more contrast or clarity is required.

History

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This document was written by R.J. Kit L., a maths student. I am not otherwise affiliated with the university, and cannot help you with related matters.

Please send me a PM on Discord @Desync#6290, a message in the WMX server, or an email to Warwick.Mathematics.Exchange@gmail.com for any corrections. (If this document somehow manages to persist for more than a few years, these contact details might be out of date, depending on the maintainers. Please check the most recently updated version you can find.)

If you found this guide helpful and want to support me, you can buy me a coffee!

(Direct link for if hyperlinks are not supported on your device/reader: ko-fi.com/desync.)

*Storing dates in big-endian format is clearly the superior option, as sorting dates lexicographically will also sort dates chronologically, which is a property that little and middle-endian date formats do not share. See ISO-8601 for more details. This footnote was made by the computer science gang.

1 Riemann Integration

Given a function $f : [a,b] \to \mathbb{R}$, we can interpret the Riemann integral as the signed area enclosed between the graph of f and the x-axis.

We formalise this notion with the use of Darboux sums.

1.1 Partitions

We begin by introducing some terminology for intervals and partitions.

An interval $[a,b]$ is *non-trivial* if $a < b$. Two intervals I and J are *almost-disjoint* if they have at most one common point – that is, $|I \cap J| = 1$.

Let $I = [a,b]$ be a non-trivial closed interval over $\mathbb{R}$. A *partition* of I is a collection $\{I_1, \dots, I_n\}$ of almost-disjoint non-trivial intervals called *subintervals* with union $\bigcup_i I_i = I$.

Note that, because a partition must be almost-disjoint, but union to the total interval, it is entirely determined by the set of points $\{x_i\}_{i=0}^n$ satisfying

$$a = x_0 < x_1 < \cdots < x_n = b$$

corresponding to the endpoints of the component intervals.

Given a partition of $P = \{I_1, \dots, I_n\}$ of an interval $I = [a,b]$, we define the quantities:

$$M := \sup_I f \qquad m := \inf_I f$$
$$M_k := \sup_{I_k} f \qquad m_k := \inf_{I_k} f$$

Note that if f is unbounded, then some of these quantities will be infinite.

Given a function $f : [a,b] \to \mathbb{R}$ and a partition $P = \{I_1, \dots, I_n\}$ of $[a,b]$, we define the *upper Darboux sum* of f with respect to P as:

$$U(f,P) := \sum_{k=1}^{n} M_k |I_k|$$

and similarly, the *lower Darboux sum* of f with respect to P as:

$$L(f,P) := \sum_{k=1}^{n} m_k |I_k|$$

Intuitively, the upper (lower) Darboux sum under-approximates (resp. over-approximates) the area bounded by f and the x-axis by approximating the area A under the function over each subinterval I_k as a rectangle with height $\inf_{x \in I_k} f(x)$ (resp. sup).

This gives, by construction,

$$m(b-a) \le L(f,P) \le A \le U(f,P) \le M(b-a)$$

where the outer terms are the Darboux sums using the whole interval as a partition. If A fails to exist, then the inequality is simply

$$m(b-a) \le L(f,P) \le U(f,P) \le M(b-a)$$

Denote by $\mathscr{P}$ the set of all partitions of $[a,b]$. Then we define the *upper Darboux integral* of f by:

$$U(f) := \inf_{p \in \mathscr{P}} U(f,P)$$

and similarly, the *lower Darboux integral*

$$L(f) \coloneqq \sup_{p\in\mathscr{P}} L(f,P)$$

We say that a bounded function $f : [a,b] \to \mathbb{R}$ is *Darboux integrable* or *Riemann integrable*[*] if $U(f) = L(f)$, and we define the Riemann integral $\int_a^b f(x)\,dx$ by

$$\int_a^b f(x)\,dx \coloneqq U(f) = L(f)$$

noting that unbounded functions are not Riemann integrable by this definition, as one of the sums will be infinite.

1.2 Refinements

A partition $Q = \{J_1,\ldots,J_\ell\}$ of $[a,b]$ is a *refinement* of a partition $P = \{I_1,\ldots,I_n\}$ if every subinterval $I_k \in P$ is the union of intervals $J_k \in Q$.

Using our alternative characterisation of partitions as collection of interval endpoints, $Q = \{y_0,\ldots,y_\ell\}$ is a refinement of $P = \{x_0,\ldots,x_n\}$ if and only if $P \subseteq Q$.

Note that this means that every partition is a refinement of itself. It is also possible for neither of two partitions to be refinements of each other.

Theorem 1.1. *Let $f : I \to \mathbb{R}$ be a bounded function, and P,Q be partitions of I, with Q a refinement of P. Then,*

$$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P)$$

That is, refining a partition gives a better approximation to the desired area.

Theorem 1.2. *Let $f : I \to \mathbb{R}$ be a bounded function, and P,Q be arbitrary partitions of I. Then,*

$$L(f,P) \le U(f,Q)$$

Corollary 1.2.1. *Let $f : I \to \mathbb{R}$ be a bounded function. Then,*

$$L(f) \le U(f)$$

Theorem 1.3. *Let $f : I \to \mathbb{R}$ be a bounded function. Then, f is Riemann integrable if and only if for every $\epsilon > 0$, there exists a partition P of I such that*

$$U(f,P) - L(f,P) < \epsilon$$

We give an alternative characterisation of Riemann integrability, through the use of sequences.

Theorem 1.4. *Let $f : I \to \mathbb{R}$ be a bounded function. Then, f is Riemann integrable if and only if there exists a sequence of partitions P_n such that*

$$\lim_{n\to\infty} U(f,P_n) - L(f,P_n) < \epsilon$$

[*] So, the notes call all of the above sums "Riemann sums", but general Riemann sums take the height of the function at arbitrary points within each subinterval, often the leftmost and rightmost points, defining the *left* and *right* Riemann sums, while Darboux sums take the infimum and supremum instead.

Unlike upper and lower Darboux sums, left and right Riemann sums do not obey a nice inequality, but in the limit, the two notions agree, and indeed, a function is Darboux integrable if and only if it is Riemann integrable, and the values of the two integrals agree whenever they exist.

To mark the distinction, and for consistency with most other sources, "Darboux" is used above when describing these sums, but due to their equivalence, I will continue to use "Riemann" when describing these integrals.

1.3 Continuity & Integrability

We recall that a function $f : I \to \mathbb{R}$ is *continuous* at $x \in I$ if for every $\epsilon > 0$, there exists a $\delta(x,\epsilon) > 0$ such that for all $y \in I$,

$$|x - y| < \delta \to |f(y) - f(x)| < \epsilon$$

noting that we may only talk about one-sided continuity for the endpoints of I. Then, we say that f is continuous on I if f is continuous at every $x \in I$, with the case of endpoints understood as one-sided continuity.

Note that, in this definition, δ is a function of both x and ϵ. If we restrict δ to be a function of ϵ, we obtain the definition of *uniform continuity*:

Given a function $f : I \to \mathbb{R}$, we say f is *uniformly continuous* if for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for all $x,y \in I$, we have,

$$|x - y| < \delta \to |f(y) - f(x)| < \epsilon$$

The difference here is that, in uniform continuity there is a globally applicable δ that depends on only ϵ, while in (ordinary) continuity there is only a locally applicable δ that depends on both ϵ and x. Thus, continuity is a local property of a function – that is, whether a function f is continuous or not at a particular point x can be determined by looking only at the values of f in an arbitrarily small neighbourhood of x. Conversely, uniform continuity is a global property of a function.

Uniform continuity is a stronger continuity condition than continuity: that is, a function that is uniformly continuous is continuous, but a function that is continuous is not necessarily uniformly continuous.

In particular, functions that are unbounded on a bounded domain cannot be uniformly continuous. For instance, the function $f : (0,1) \to \mathbb{R}$ defined by $x \mapsto \frac{1}{x}$ approaches infinity at an increasing rate as x approaches the origin, so it is not possible to find a δ independent of x that satisfies the definition of continuity.

Functions that have gradients that become unbounded on an infinite domain also cannot be uniformly continuous. For instance, $f : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto e^x$ is continuous everywhere, but its gradient becomes arbitrarily large, so it is possible to find arbitrarily small intervals in which f varies by more than ϵ.

Theorem 1.5. *Let I be a compact subset of $\mathbb{R}$ (i.e. a closed interval), and suppose $f : I \to \mathbb{R}$ is continuous. Then, f is uniformly continuous.*

We now give some sufficient (but not necessary) conditions for Riemann integrability.

Theorem 1.6. *If $f : [a,b] \to \mathbb{R}$ is continuous, then it is Riemann integrable.*

Theorem 1.7. *If $f : [a,b] \to \mathbb{R}$ is monotonic, then it is Riemann integrable.*

1.4 Algebra of Integrals

Theorem 1.8. *Let $f,g : [a,b] \to \mathbb{R}$ be Riemann integrable functions, and let $c \in \mathbb{R}$. Then, $f + g$ and cf are Riemann integrable, and satisfy,*

$$\int_a^b cf = c\int_a^b f, \qquad \int_a^b (f + g) = \int_a^b f + \int_a^b g$$

Theorem 1.9. *Let $f,g : [a,b] \to \mathbb{R}$ be integrable functions such that $f(x) \leq g(x)$ for all $x \in [a,b]$. Then,*

$$\int_a^b f \leq \int_a^b g$$

Corollary 1.9.1. *If $f : [a,b] \to \mathbb{R}$ is integrable, then,*

$$m(b-a) \leq \int_a^b f \leq M(b-a)$$

Corollary 1.9.2. *If $f : [a,b] \to \mathbb{R}$ is continuous, then there exists $c \in [a,b]$ such that*

$$f(c) = \frac{1}{b-a}\int_a^b f$$

Theorem 1.10. *If $f : [a,b] \to \mathbb{R}$ is integrable, then $|f|$ is integrable, and,*

$$\left|\int_a^b f\right| \leq \int_a^b |f|$$

Theorem 1.11. *Let $f : [a,b] \to \mathbb{R}$ and $c \in (a,b)$. Then, f is integrable on $[a,b]$ if and only if it is integrable on $[a,c]$ and $[c,b]$, and moreover,*

$$\int_a^c f + \int_c^b f = \int_a^b f$$

Theorem 1.12. *If $f : [a,b] \to \mathbb{R}$ is integrable and $g : \mathbb{R} \to \mathbb{R}$ is continuous, then $g \circ f$ is integrable.*

Note that the composition of two integrable functions is not necessarily integrable.

Theorem 1.13. *If $f,g : [a,b] \to \mathbb{R}$ are integrable, then the product function fg is integrable, and, if additionally $\frac{1}{g}$ is bounded, then $\frac{f}{g}$ is integrable.*

1.5 Fundamental Theorem of Calculus

The fundamental theorem of calculus links the notions of differentiation and integration together as inverses.

Theorem 1.14 (FTC I)**.** *Let $f : [a,b] \to \mathbb{R}$ be continuous, and define $F : [a,b] \to \mathbb{R}$ by*

$$F(x) = \int_a^x f(t)\,dt$$

Then, F is uniformly continuous on $[a,b]$ and differentiable on (a,b), with $F'(x) = f(x)$ for all $x \in (a,b)$, and we say that F is an antiderivative of f.

Equivalently,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Proof. We compute the derivative of $F(x)$ from the definition:

$$\begin{aligned}
F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\left[\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right] \\
&= \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt
\end{aligned}$$

By the mean value theorem for integrals, there exists $c \in [x,x+h]$ such that $f(c) \cdot h = \int_x^{x+h} f(t)\,dt$, so,

$$= \lim_{h \to 0} f(c)$$

$c \in [x,x+h]$, so by the sandwich theorem,

$$= f(x)$$

∎

Corollary 1.14.1. *Let $f : [a,b] \to \mathbb{R}$ be continuous with antiderivative F on $[a,b]$. Then,*

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

Theorem 1.15 (FTC II)**.** *Let $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ with continuous antiderivative F on (a,b). Then,*

$$\int_a^b f(x)\,dx = F(b) - f(a)$$

Unlike in the corollary above, FTC II does not require continuity of f over $[a,b]$, and is thus a slightly stronger result.

Proof. We wish to show

$$L(f,P) \leq F(b) - F(a) \leq U(f,P)$$

for every partition P of $[a,b]$. By taking a supremum on the left, and infimum on the right, we obtain $L(f) \leq F(b) - F(a) \leq U(f)$, and since f is integrable, both sides reduce to equalities.

Now, consider any partition $P = \{a = x_0, x_1, \ldots, x_{n-1}, x_n = b\}$. On every interval $I_k = [x_{k-1}, x_k]$, for every $c_k \in (x_{k-1}, x_k)$ we have,

$$\inf_{I_k} f(x)(x_k - x_{k-1}) \leq f(c_k)(x_k - x_{k-1}) \leq \sup_{I_k} f(x)(x_k - x_{k-1})$$

As F is continuous on $[x_{k-1}, x_k]$ and differentiable on (x_{k-1}, x_k), by the mean value theorem there exists c_k such that $F(x_k) - F(x_{k-1}) = f(c_k)(x_k - x_{k-1})$, so we have,

$$\inf_{I_k} f(x)(x_k - x_{k-1}) \leq F(x_k) - F(x_{k-1}) \leq \sup_{I_k} f(x)(x_k - x_{k-1})$$

Summing over $k = 1$ to n, we have,

$$L(f,P) \leq \sum_{k=1}^{n} F(x_k) - F(x_{k-1}) \leq U(f,P)$$

This sum telescopes to,

$$\begin{aligned} L(f,P) &\leq F(x_0) - F(x_n) \leq U(f,P) \\ L(f,P) &\leq F(b) - F(a) \quad \leq U(f,P) \end{aligned}$$

thus proving the result. ∎

Theorem 1.16. *If $f : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$ and is continuous from the right at a, then,*

$$\lim_{h\to 0^+} \frac{1}{h} \int_a^{a+h} f(t)\,dt = f(a)$$

and similarly, if f is continuous from the left at b,

$$\lim_{h\to 0^+} \frac{1}{h} \int_{b-h}^{b} f(t)\,dt = f(b)$$

More generally, if (I_h) is a sequence of intervals such that $|I_h| \to 0$, $x \in I_h$ for all h, and f is continuous at x, then,

$$\lim_{h\to 0} \frac{1}{|I_h|} \int_{I_h} f(t)\,dt = f(x)$$

Integration by parts and u-substitution are both consequences of the fundamental theorem of calculus:

Theorem 1.17 (IBP)**.** *If $f,g : [a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and differentiable on (a,b) such that f',g' are integrable on $[a,b]$, then,*

$$\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,dx$$

Theorem 1.18 (u-sub)**.** *Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$ (understood as one-sided differentiability at the endpoints) such that f' is integrable on $[a,b]$, and let g be continuous on $f\big([a,b]\big)$. Then,*

$$\int_a^b g\big(f(x)\big)f'(x)\,dx = \int_{f(a)}^{f(b)} g(u)\,du$$

1.6 Improper Integration

So far, we have only defined Riemann integrals for bounded functions over bounded intervals. Now, we extend this definition to include unbounded functions and/or unbounded intervals using limits. This extension is called an *improper Riemann integral.*

Let $f : (a,b] \to \mathbb{R}$ be Riemann integrable over every interval $[c,b] \subset (a,b]$. Then, the improper intergral of f on $[a,b]$ is defined as,

$$\int_a^b f(x)\,dx \coloneqq \lim_{\epsilon\to 0^+} \int_{a+\epsilon}^{b} f(x)\,dx$$

If this limit is finite, then the improper integral *converges*, *diverging* otherwise.

Similarly, if $f : [a,b) \to \mathbb{R}$ is integrable over every interval $[a,c] \subset [a,b)$, then the improper intergral of f on $[a,b]$ is defined as,

$$\int_a^b f(x)\,dx \coloneqq \lim_{\epsilon\to 0^+} \int_a^{b-\epsilon} f(x)\,dx$$

We can also define an inproper integral if the function is unbounded at an interior point c.

Let $f : [a,b] \setminus \{c\} \to \mathbb{R}$ be a function integrable on any closed interval not containing $c \in [a,b]$. That is, f is integrable on $[a,c-\epsilon_1]$ and $[c+\epsilon_2,b]$ for all sufficiently small $\epsilon_1,\epsilon_2 > 0$. Then,

$$\int_a^b f(x)\,dx \coloneqq \lim_{\epsilon_1\to 0^+} \int_a^{c-\epsilon_1} f(x)\,dx + \lim_{\epsilon_2\to 0^+} \int_{c+\epsilon_2}^{b} f(x)\,dx$$

For unbounded domains of integration, we take a limit of ordinary integrals:

If $f : [a,\infty) \to \mathbb{R}$ is integrable for every interval $[a,y] \subset [a,\infty)$, then,

$$\int_a^\infty f(x)\,dx := \lim_{y\to\infty} \int_a^y f(x)\,dx$$

Similarly, if $f : (-\infty,b] \to \mathbb{R}$ is integrable for every interval $[y,b] \subset (-\infty,b]$, then,

$$\int_{-\infty}^b f(x)\,dx := \lim_{y\to-\infty} \int_y^b f(x)\,dx$$

and if $f : \mathbb{R} \to \mathbb{R}$ is integrable on every bounded interval $[a,b]$, then,

$$\int_{-\infty}^\infty f(x)\,dx := \lim_{a\to-\infty} \int_a^c f(x)\,dx + \lim_{b\to\infty} \int_c^b f(x)\,dx$$

for any $c \in \mathbb{R}$.

The space of functions that are improperly Riemann integrable forms a linear space: that is, if f and g are improperly integrable on the same domain, then $\alpha f + \beta g$ is also improperly integrable over the same domain for any $\alpha,\beta \in \mathbb{R}$.

Theorem 1.19 (Absolute Comparison Test). *Let $f : [a,\infty) \to \mathbb{R}$ be integrable on $[a,b]$ for every $b > a$. If $\int_a^\infty |f| < \infty$, then $\int_a^\infty f$ converges, and we say that $\int_a^\infty f$ is absolutely convergent.*

Moreover, if $g : [a,\infty) \to [0,\infty)$ is a function such that $|f| \le g$ and $\int_a^\infty g < \infty$, then $\int_a^\infty f$ is absolutely convergent.

2 Sequences and Series of Functions

2.1 Convergence

Let $(f_n)_{n=0}^\infty$ be a sequence of functions $f_n : \Omega \to \mathbb{R}$. We say that (f_n) *converges pointwise* to $f : \Omega \to \mathbb{R}$ if for every $x \in \Omega$,

$$\lim_{n\to\infty} f_n(x) = f(x)$$

and we denote this relation by $f_n \to f$.

Intuitively, a sequence (f_n) of functions converges pointwise to a function f if, when we fix any choice of input value x, the resulting sequence of output terms $\big(f_n(x)\big)_{n=0}^\infty$ (which is just a sequence of real numbers) converges to the output value $f(x)$ in the usual sense.

Note that the pointwise limit of a sequence of continuous functions is not necessarily continuous.

Let $(f_n)_{n=0}^\infty$ be a sequence of functions $f_n : \Omega \to \mathbb{R}$. We say that (f_n) *converges uniformly* to $f : \Omega \to \mathbb{R}$ if for any $\epsilon > 0$, there exists $N(\epsilon)$ such that $|f_n(x) - f(x)| < \epsilon$ for every $x \in \Omega$ and every $n > N(\epsilon)$, and we denote this relation by $f_n \rightrightarrows f$.

Uniform convergence is to pointwise convergence what uniform continuity is to ordinary continuity: in uniform convergence, N depends only on ϵ, and not on x, while in pointwise continuity, we began by fixing a value of x.

To simplify notation, we define the ℓ^∞, *supremum* or *Chebyshev* norm by:

$$\|f\|_\infty := \sup_{x\in\Omega} |f(x)|$$

Using this, we can simplify the definition of uniform convergence to:

$$f_n \rightrightarrows f := \forall\epsilon > 0, \exists N(\epsilon), \forall n > N(e) : \|f_n - f\|_\infty < \epsilon.$$

Theorem 2.1. *Uniform convergence implies pointwise convergence, but not the converse.*

A sequence (f_n) of functions in Ω is *uniformly Cauchy* if for every $\epsilon > 0$, there exists $N(\epsilon)$ such that $\|f_n - f_m\|_\infty < \epsilon$ for all $n,m > N(\epsilon)$.

Theorem 2.2. *A sequence (f_n) of functions is uniformly convergent if and only if it is uniformly Cauchy.*

Theorem 2.3. *If a sequence of continuous functions (f_n) in Ω converges uniformly to a function $f : \Omega \to \mathbb{R}$, then f is continuous.*

The space of bounded continuous functions on a space Ω is denoted $C_b(\Omega)$.

Theorem 2.4. *$(C_b(\Omega),\|\cdot\|_\infty)$ is a complete space: that is, every Cauchy sequence converges to a continuous bounded function, etc.*

Theorem 2.5. *Let (f_n) be a sequence of Riemann integrable functions $f_n : [a,b] \to \mathbb{R}$ that converges uniformly to a function $f : [a,b] \to \mathbb{R}$. Then, f is Riemann integrable and $\int f_n \to \int f$.*

Uniform convergence and differentiation do not interact as nicely. There are examples of sequences of differentiable functions (f_n), with (f_n) converging uniformly to f, but (f_n') does not converge to f' (or f' may fail to exist). This also does not hold even if the sequence is of infinitely differentiable functions.

2.2 Multivariate Continuity

We now introduce definitions of (uniform) continuity of functions defined over subsets of $\mathbb{R}^2$.

We write $C^k(\Omega)$ to denote the space of functions that are k times continuously differentiable over Ω, and $C^\infty(\Omega)$ for the space of functions infinitely differentiable over Ω, also called functions that are *smooth* over Ω.

A function $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous at $x \in \Omega$ if for every $\epsilon > 0$, there exists $\delta(x,\epsilon) > 0$ such that for all $y \in \Omega$,

$$\|x - y\| < \delta \to |f(y) - f(x)| < \epsilon$$

Similarly, a function $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is uniformly continuous if for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for all $x,y \in \Omega$,

$$\|x - y\| < \delta \to |f(y) - f(x)| < \epsilon$$

and again, the difference here is that δ is independent of x.

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^2$ be closed and bounded. Then, any continuous function $f : \Omega \to \mathbb{R}$ is furthermore uniformly continuous.*

Theorem 2.7. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be continuous. Define $I : [c,d] \to \mathbb{R}$ by*

$$I(t) := \int_a^b f(x,t)\, dx$$

Then, I is continuous.

Theorem 2.8 (Leibniz Integral Rule)**.** *Let $f, \frac{\partial f}{\partial t}$ be continuous functions on $[a,b] \times [c,d]$. Then, for any $t \in (c,d)$,*

$$\frac{d}{dt}\int_a^b f(x,t)\, dx = \int_a^b \frac{\partial f}{\partial t}(x,t)\, dx$$

Theorem 2.9 (Fubini's Theorem for Continuous Functions)**.** *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be continuous. Then,*

$$\int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

Theorem 2.10. *Let (f_n) be a sequence of C^1 functions on $[a,b]$, and suppose $f_n \to f$ (pointwise), and $f' \rightrightarrows g$ (uniformly). Then, f is C^1 and $g = f'$ (that is, $f_n' \rightrightarrows f'$).*

2.3 Series

We now define the notions of pointwise and uniform convergence for series of functions.

Let (f_k) be a sequence of functions $f_k : \Omega \to \mathbb{R}$, and let (S_n) be the sequence of partial sums of (f_k), with $S_n : \Omega \to \mathbb{R}$ defined by

$$S_n(x) = \sum_{k=1}^{n} f_k(x)$$

Then, the series

$$\sum_{k=1}^{\infty} f_k(x)$$

is said to converge pointwise to $S : \Omega \to \mathbb{R}$ in Ω if $S_n \to S$ pointwise in Ω, and to converge uniformly to S in Ω if $S_n \rightrightarrows S$ uniformly on Ω.

Theorem 2.11. *If (f_k) is a series of integrable functions $f_k : [a,b] \to \mathbb{R}$, and S_n converges uniformly, then $\sum_{k=1}^{\infty} f_k$ is Riemann integrable, and,*

$$\int \sum_{k=1}^{\infty} f_k = \sum_{k=1}^{\infty} \int f_k$$

Theorem 2.12. *Let (f_k) be a sequence of C^1 functions $f_k : [a,b] \to \mathbb{R}$ such that S_n converges pointwise, and suppose that $\sum_{k=1}^{n} f_k'$ converges uniformly. Then,*

$$\left(\sum_{k=1}^{\infty} f_k \right)' = \sum_{k=1}^{\infty} f_k'$$

That is, the series is differentiable and can be differentiated term-by-term.

Theorem 2.13 (Weierstrass M-test)**.** *Let (f_k) be a sequence of functions $f_k : \Omega \to \mathbb{R}$, and suppose that exists a sequence (M_k) of non-negative reals such that*

- *$|f_k(x)| \leq M_k$ for all $k \in \mathbb{N}$ and all $x \in \Omega$;*
- *$\sum_{k=1}^{\infty} M_k$ converges.*

Then, the series $\sum_{k=1}^{\infty} f_n(x)$ converges absolutely and uniformly on Ω.

Proof. We show that the partial sums $S_n = \sum_{k=1}^{n} f_k(x)$ is uniformly Cauchy. Now, since $\sum_{k=1}^{\infty} M_k$ converges, given $\varepsilon > 0$, there exists N such that

$$\sum_{k=m+1}^{n} M_k < \varepsilon$$

for all $m,n > N$. Now,

$$|S_n(x) - S_m(x)| = \left| \sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{m} f_k(x) \right|$$

$$
\begin{aligned}
&= \left| \sum_{k=m+1}^{n} f_k(x) \right| \\
&\leq \sum_{k=m+1}^{n} |f_k(x)| \\
&\leq \sum_{k=m+1}^{n} M_k \\
&< \varepsilon
\end{aligned}
$$

■

3 Complex Analysis

We quickly revisit some basic properties of the complex numbers.

The set of complex numbers $\mathbb{C}$ is given by

$$\mathbb{C} = \{x + iy : x,y \in \mathbb{R}\}$$

where i is the imaginary unit, satisfying $i^2 = -1$.

For a complex number $z = x + iy$, we denote

- the real component of z by $\Re(z) = x$;
- the complex component of z by $\Im(z) = y$;
- the modulus or norm of z by $|z| = \sqrt{x^2 + y^2}$;
- the complex conjugate of z by $\bar{z} = x - iy$.

Theorem 3.1. *The following statements hold for any complex numbers $z,w \in \mathbb{C}$.*

- $\bar{\bar{z}} = z$;
- $\overline{z + w} = \bar{z} + \bar{w}$;
- $\overline{zw} = \bar{z}\bar{w}$;
- $|\bar{z}| = |z|$;
- $|z|^2 = z\bar{z}$

A sequence $(z_n)_{n=1}^{\infty} \subset \mathbb{C}$ converges to a complex number $z \in \mathbb{C}$ if $\lim_{n\to\infty} |z_n - z| = 0$. That is, if for every $\epsilon > 0$, there exists $N > 0$ such that $|z_n - z| < \epsilon$ for all $n > N$.

A set $\Omega \subseteq \mathbb{C}$ is *open* if for every $x \in \Omega$, there exists $r > 0$ such that $\mathbb{B}_r(x) \subset \Omega$, and a set $\Omega \subseteq \mathbb{C}$ is closed if $\Omega^c = \mathbb{C} \setminus \Omega$ is open.

A set $K \subset \mathbb{C}$ is *sequentially compact* if every sequence $(x_j)_{j=1}^{\infty} \subset K$ has a convergent subsequence $(x_{j(\ell)})_{\ell=1}^{\infty}$ whose limit is in K.

A function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ is continuous at $z_0 \in \Omega$ if for every $\epsilon > 0$, there exists a $\delta(x,\epsilon) > 0$ such that for all $z \in \Omega$,

$$|z - z_0| < \delta \to |f(z) - f(z_0)| < \epsilon$$

This definition is identical to that of continuity for real functions, but with $|\cdot|$ now being a norm on $\mathbb{C}$ rather than $\mathbb{R}$, and in fact, coincides with the definition of continuity for functions $\mathbb{R}^2 \to \mathbb{R}^2$.

3.1 Complex Differentiability

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a point p if the limit

$$\lim_{h\to 0} \frac{f(p+h) - f(p)}{h}$$

exists, and this limit is the value of the derivative.

In contrast, a function $f : \mathbb{R}^n \to \mathbb{R}^k$ is differentiable at a point p if there exists a linear map $Df \in L(\mathbb{R}^n; \mathbb{R}^k)$ such that

$$\lim_{h\to 0} \frac{|f(p+h) - f(p) - Df(p)h|}{|h|} = 0$$

and this linear map Df is the value of the derivative.

We use this definition because for $k > 1$, there is no well-defined notion of division of vectors.

However, unlike in $\mathbb{R}^2$, $\mathbb{C}$ does have a notion of division we can use, so we can return to the original definition of differentiability, and so, differentiability for functions $\mathbb{C} \to \mathbb{C}$ is distinct from (and in many ways, more well-behaved than) functions $\mathbb{R}^2 \to \mathbb{R}^2$.

Let $\Omega \subset \mathbb{C}$ be an open set. A function $f : \mathbb{C} \to \mathbb{C}$ is *complex differentiable* at a point $z_0 \in \Omega$ if the limit

$$\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

exists, and this limit is the value of the derivative.

However, here, h is a complex number, so there are many ways we could send h to 0. If this limit exists, then its value should be independent of the path taken. We will write $f(x,y)$ as $u(x,y) + iv(x,y)$ to separate out the real and imaginary components.

Now, consider approaching along the real axis. We have,

$$\lim_{\substack{h\to 0 \\ h\in\mathbb{R}}} \frac{f(z_0+h) - f(z_0)}{h} = \frac{\partial f}{\partial x}(z_0)$$

while approaching along the imaginary axis gives,

$$\lim_{\substack{h\to 0 \\ h\in\mathbb{C}}} \frac{f(z_0+h) - f(z_0)}{h} = \frac{1}{i}\frac{\partial f}{\partial y}(z_0)$$

These values should be equal, and so,

$$\begin{aligned} i\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial y} \\ i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) &= \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} \\ -\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} \end{aligned}$$

Equating the real and imaginary components, we have,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

or more compactly,

$$u_x = v_y, \qquad u_y = -v_x$$

These are the *Cauchy-Riemann equations.* For a complex derivative to exist, these equations must be satisfied.

Moreover, if $f : \mathbb{C} \to \mathbb{C}$ is a function that is differentiable when regarded as a function $f : \mathbb{R}^2 \to \mathbb{R}$, then f is complex differentiable if and only if the Cauchy-Riemann equations hold.

This means that if the components u and v are real-differentiable functions of two real variables, then $u + iv$ is a complex-valued real-differentiable function, and is furthermore complex-differentiable if and only if the Cauchy-Riemann equations hold. We can also replace the requirement that u and v are differentiable with the requirement that all partial derivatives of u and v are continuous (as this implies that u and v are real-differentiable).

Example. Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by $z \mapsto z^2$. u and v are clearly continuous, so f is real-differentiable.

$$\begin{aligned} f(z) &= (x + iy)^2 \\ &= x^2 - y^2 + 2ixy \end{aligned}$$

so,

$$u(x,y) = x^2 - y^2, \quad v(x,y) = 2xy$$

with partial derivatives

$$u_x = 2x, \quad u_y = -2y, \quad v_x = 2y, \quad v_y = 2x$$

satisfying the Cauchy-Riemann equations, so f is also complex-differentiable.

A function $f : \Omega \to \mathbb{C}$ is *holomorphic* at $z_0 \in \Omega$ if there exists a neighbourhood $U \subseteq \Omega$ of z_0 such that f is complex-differentiable at all $z \in U$.

f is holomorphic in Ω if f is holomorphic at all $z \in \Omega$, and we say that f is *entire* if it is holomorphic on the whole of $\mathbb{C}$.

A general function $f : A \to B$ is *analytic* at a point if it is given locally by a convergent power series at that point. That is, f is analytic at x_0 if the Taylor series centred at x_0 converges pointwise to $f(x)$ for every x in a neighbourhood $U \subseteq B$. Note that a function may be complex-differentiable at a point, but not necessarily analytic.

Earlier, we mentioned that complex functions are sometimes more well-behaved than real functions; it turns out that a complex-valued function is analytic if and only if it is holomorphic, so the terms are sometimes used interchangably in the context of complex analysis.

Theorem 3.2 (Algebra of Complex Derivatives)**.** *Let $f,g : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ be complex-differentiable functions. Then,*

$$(f+g)' = f' + g', \qquad (fg)' = f'g + fg', \qquad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}, \qquad (f(g))' = f'(g)g'$$

(assuming that $g \neq 0$ in the third expression, and that the domains and codomains are appropriate in the fourth).

Theorem 3.3. *The function $f : \mathbb{C} \to \mathbb{C}$ defined by $z \mapsto z^n$ is entire for all $n \in \mathbb{N}$, and $f'(z) = nz^{n-1}$.*

3.2 Power Series

We define the notions of convergence of series in $\mathbb{C}$ similarly to that of series in $\mathbb{R}$.

Let $(a_n)_{n=0}^{\infty}$ be a sequence of complex numbers $a_n \in \mathbb{C}$. The series $\sum_{n=0}^{\infty} a_n$ is *convergent* if the sequence of partial sums $S_k = \sum_{n=0}^{k} a_n$ is convergent in C, and is *absolutely convergent* if the series $\sum_{n=0}^{\infty} |a_n|$ is convergent in C.

The geometric series $\sum_{n=0}^{\infty} a_n$ is absolutely convergent if and only if $|z| < 1$ with limit

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

and partial sums

$$S_k = \frac{1 - z^{k+1}}{1-z}$$

Theorem 3.4 (Ratio Test). *Let $(a_n)_{n=0}^{\infty}$ be a sequence of complex numbers $a_n \in \mathbb{C}$ with $a_n \neq 0$ for all sufficiently large n, and define the quantity,*

$$L := \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Then,

- *if $L < 1$, then the series $\sum_{n=0}^{\infty} a_n$ converges absolutely;*
- *if $L > 1$, then the series $\sum_{n=0}^{\infty} a_n$ diverges;*
- *if $L = 1$ or the limit fails to exist, then the test is inconclusive.*

Using suprema and infima, we can strengthen this test: define the quantities,

$$R := \limsup \left| \frac{a_{n+1}}{a_n} \right|, \qquad r := \liminf \left| \frac{a_{n+1}}{a_n} \right|$$

Then,

- *if $R < 1$, then the series $\sum_{n=0}^{\infty} a_n$ converges absolutely;*
- *if $r > 1$, then the series $\sum_{n=0}^{\infty} a_n$ diverges;*
- *if $\left| \frac{a_{n+1}}{a_n} \right| > 1$ for all sufficiently large n, then the series $\sum_{n=0}^{\infty} a_n$ also diverges;*
- *otherwise, the test is inconclusive.*

Theorem 3.5. *Consider $\sum_{n=0}^{\infty} a_n$ and define the quantity,*

$$r := \limsup \sqrt[n]{|a_n|}$$

- *If $r < 1$, then the series converges;*
- *If $r > 1$, then the series diverges;*
- *If $r = 1$, then the test is inconclusive.*

The root test is stronger than the ratio test: whenever the ratio test determines the convergence or divergence of an infinite series, the root test does too, but not the converse.

Theorem 3.6. *Given any sequence $(a_n)_{n=0}^{\infty}$, there exists $R \in [0,\infty]$ such that*

$$\sum_{n=0}^{\infty} a_n z^n$$

converges for all $|z| < R$ and diverges for $|z| > R$.

More specifically, this value is given by,

$$R = \frac{1}{\limsup \sqrt[n]{a_n}}$$

Theorem 3.7. *Let $a_n \neq 0$ for all $n \geq N$ and suppose that $\lim_{n\to 0} \frac{|a_{n+1}|}{|a_n|}$ exists. Then, $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence,*

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$$

Theorem 3.8. *Suppose a series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R. Then, for all $|z| < R$, the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is differentiable and,*

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

That is, the derivative may be computed term by term.

Corollary 3.8.1. *Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R > 0$. Then, the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is smooth (infinitely differentiable), and moreover,*

$$\frac{f^{(n)}(0)}{n!} = a_n$$

for all $n \in \mathbb{N}_0$.

Theorem 3.9. *Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R > 0$. Then, for every $r < R$, the sequence of functions,*

$$f_k \coloneqq \sum_{n=0}^{k} a_n z^n$$

converges uniformly in $|z| \leq r$.

3.3 The Complex Exponential

In this section, I write $\exp(z)$ instead of e^z to emphasise that these power series are definitions and not theorems, unlike the case for the real exponential.

We define the following power series for $z \in \mathbb{C}$.

$$\begin{aligned}
\exp(z) &\coloneqq \sum_{n=0}^{\infty} \frac{1}{n!} z^n \\
&= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \\
\cos(z) &\coloneqq \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} \\
&= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots
\end{aligned}$$

$$
\begin{aligned}
\cosh(z) &:= \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} \\
&= 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots \\
\sin(z) &:= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \\
&= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \\
\sinh(z) &:= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1} \\
&= z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots
\end{aligned}
$$

These functions are entire, converging for any $z \in \mathbb{C}$.

Theorem 3.10. *The following identities hold for all $z \in \mathbb{C}$:*

$$
\begin{aligned}
\cos(z) &= \frac{\exp(iz) + \exp(-iz)}{2}, & \cosh(z) &= \frac{\exp(z) + \exp(-z)}{2}, \\
\sin(z) &= \frac{\exp(iz) - \exp(-iz)}{2i}, & \sinh(z) &= \frac{\exp(z) - \exp(-z)}{2}
\end{aligned}
$$

$$
\cos(iz) = \cosh(z), \qquad \cosh(iz) = \cos(z), \qquad \sin(iz) = i\sinh(z), \qquad \sinh(iz) = i\sin(z)
$$

Theorem 3.11. *The complex exponential function* $\exp(z)$ *satisfies the following:*

- *(Characteristic Property of the Exponential)* $\exp(z+w) = \exp(z)\exp(e)$ *for all* $z,w \in \mathbb{C}$, $\exp(1) = e$;
- $\exp(z) \neq 0$ *for all* $z \in \mathbb{C}$;
- $\exp(z) = 1$ *if and only if* $z = 2k\pi i$ *with* $k \in \mathbb{Z}$;
- $\exp(z) = -1$ *if and only if* $z = (2k+1)\pi i$ *with* $k \in \mathbb{Z}$.

The third property implies that $\exp(z + w) = \exp(z)$ if and only if $w = 2k\pi i$, $k \in \mathbb{Z}$, so the exponential function is periodic along the imaginary axis with period 2π.

3.4 The Complex Logarithm

Every complex number $z = x + iy \in \mathbb{C} \setminus \{0\}$ can be written as $re^{i\theta}$, where r is the *modulus* of z, $|z|$, and θ is the *phase* of z – the angle that the vector rooted at the origin pointing to z makes with the positive real axis, measured counterclockwise. Note that for $z = 0$, this angle is undefined, and in any other case, is unique only up to factors of 2π.

We define the multivalued *argument* function $\arg : \mathbb{C} \setminus \{0\} \to \mathcal{P}(\mathbb{R})$ by

$$
\arg(z) = \{\theta \in \mathbb{R} : z = |z|e^{iz}\}
$$

The argument function is not a function in the usual sense as the image of each input is not uniquely defined: in particular, if $\theta \in \arg(z)$, then $\theta + 2k\pi \in \arg(z)$ for all $k \in \mathbb{Z}$.

Theorem 3.12. *The argument function* $\arg(z)$ *satisfies the following:*

- $\arg(\alpha z) = \arg(z)$ *for all real* $\alpha > 0$;
- $\arg(\alpha z) = \arg(z) + \pi = \{\theta + \pi : \theta \in \arg(z)\}$ *for all real* $\alpha < 0$;

- $\arg(\bar{z}) = -\arg(z) = \{-\theta : \theta \in \arg(z)\}$;
- $\arg\left(\frac{1}{z}\right) = -\arg(z)$;
- $\arg(zw) = \arg(z) + \arg(w) = \{\theta + \phi : \theta \in \arg(z), \phi \in \arg(w)\}$.

We define the *principle value argument* function $\operatorname{Arg} : \mathbb{C} \setminus \{0\} \to (-\pi,\pi]$ by taking the angle in $\arg(z)$ that lies in the interval $(-\pi,\pi]$. Then, we have $\arg(z) = \{\operatorname{Arg}(z) + 2k\pi : k \in \mathbb{Z}\}$.

Note that the Arg function is not continuous in the entire complex plane. In particular, approaching the negative real axis from the clockwise direction yields $-\pi$, while approaching from the counterclockwise direction yields π. Making any other choice for the image of Arg leads to a similar issue along the half-line where we define the ends of the image, where the arguments will differ by 2π when approaching from different directions.

We wish to define an extension of the logarithm to the complex numbers, and to mark the distinction, we will write ln to denote the ordinary real logarithm in $\mathbb{R}_{\geq 0}$, and log to denote our complex extension. One defining characteristic of the real logarithm is that $x = \ln(y)$ if and only if $e^x = y$ – that is, the real logarithm is the inverse of the real exponential.

Since $e^z = e^{z+2k\pi i}$ for any $k \in \mathbb{Z}$, then if $w = \log(z)$, then so is $w + 2k\pi i$, so the complex logarithm is also multivalued.

Let $z,w \in \mathbb{C}$ such that $w = \log(z) = u + iv$. Then, we have,

$$\begin{aligned} z &= e^{\ln(z)} \\ &= e^w \\ &= e^{u+iv} \\ &= (e^u)e^{iv} \end{aligned}$$

But, $z = |z|e^{i\arg(z)}$, so, equating the modulus and argument, we have $e^u = |z|$, and $v = \arg(z)$, with the modulus equation in particular implying that $u = \ln|z|$.

We define the multivalued complex logarithm $\log : \mathbb{C} \setminus \{0\} \to \mathcal{P}(\mathbb{R})$ by

$$\log(z) := \ln|z| + i\arg(z)$$

again noting that $\log(z)$ is undefined for $z = 0$, as $\ln|z| = \ln(0)$ is undefined.

Theorem 3.13. *The complex logarithm function* $\log(z)$ *satisfies the following:*

- $\log(zw) = \log(z) + \log(w) \pmod{2\pi i}$;
- $\log\left(\frac{z}{w}\right) = \log(z) - \log(w) \pmod{2\pi i}$;
- $\exp(\log(z)) = z$;
- $\log(\exp(z)) = z \pmod{2\pi i}$.

We define the *principle branch logarithm* $\operatorname{Log} : \mathbb{C} \setminus \{0\} \to \mathbb{R}_{\geq 0}$ by

$$\operatorname{Log}(z) := \ln|z| + i\operatorname{Arg}(z)$$

Because the Arg function is discontinuous along the half-line $x \leq 0$, the Log function is also discontinuous along the same line: if we consider points $z = x + i\epsilon$ for $x < 0$ and sufficiently small $\epsilon > 0$, we have,

$$\lim_{\epsilon \to 0} \operatorname{Log}(x \pm i\epsilon) = \operatorname{Log}|x| \pm i\pi$$

so the function cannot be extended continuously along $\{x \leq 0\}$. This half-line is called a *branch cut*, and any definition of the principle value argument function results in such a half-line.

From the identity,

$$e^{\operatorname{Log}(z)} = z$$

we have,

$$e^{\operatorname{Log}(z)}(\operatorname{Log}(z))' = 1$$

and hence,

$$(\operatorname{Log}(z))' = \frac{1}{z}$$

With the complex extension of the natural logarithm, we can now define complex powers of complex numbers. Given $\alpha, z \in \mathbb{C}$ with $z \neq 0$, we define,

$$\begin{aligned} z^\alpha &:= e^{\operatorname{Log}(z^\alpha)} \\ &= e^{\alpha \operatorname{Log}(z)} \\ &= e^{\alpha \ln|z| + \alpha i \arg(z)} \\ &= e^{\alpha \ln|z| + \alpha i \operatorname{Arg}(z) + \alpha 2ki\pi} \\ &= e^{\alpha \ln|z| + \alpha i \operatorname{Arg}(z)} e^{\alpha 2ki\pi} \end{aligned}$$

where $k \in \mathbb{Z}$, and we see that complex powers can be multivalued. Specfically, if α is an integer, then $e^{\alpha 2ki\pi} = 1$, so there is only one value of z^α. If $\alpha = \frac{p}{q}$ is rational with $p \in \mathbb{Z}, q \in \mathbb{N}$ coprime, then

$$e^{\alpha 2ki\pi} = e^{\alpha 2(k+q)i\pi}$$

and z^α will assume q distinct values. If α is irrational, then z^α will take infinitely many values.

3.5 Complex Integration

For a function $f : [a,b] \to \mathbb{C}$, we define,

$$\int_a^b f(t)\,dt := \int_a^b \Re\big(f(t)\big)\,dt + i \int_a^b \Im\big(f(t)\big)\,dt$$

So, integrating a complex-valued function reduces to integrating two real-valued functions.

Theorem 3.14. *For every $f,g : [a,b] \to \mathbb{C}$ and every $\alpha, \beta \in \mathbb{C}$, we have,*

$$\int_a^b \alpha f(t) + \beta g(t)\,dt = \alpha \int_a^b f(t)\,dt + \beta \int_a^b g(t)\,dt$$

Theorem 3.15. *For any function $f : [a,b] \to \mathbb{C}$,*

- $$\overline{\int_a^b f(t)\,dt} = \int_a^b \overline{f(t)}\,dt$$
- $$\left|\int_a^b f(t)\,dt\right| \le \int_a^b \big|f(t)\big|\,dt$$

3.6 Contour Integrals

The previous definition of an integral is a natural extension of real integration for integrating functions $\mathbb{R} \to \mathbb{C}$, but what would it mean to integrate a function $\mathbb{C} \to \mathbb{C}$? Single integrals only make sense when evaluated along 1 dimensional curves, so there is no natural extension in this case.

Because of this, we will only consider integrals of complex-valued functions *along curves* in the complex plane called *contours*:

$$\int_\Gamma f\,dz$$

where Γ is a contour in $\mathbb{C}$. To evaluate such an integral, we begin by parametrising Γ by a function $\gamma : [a,b] \to \mathbb{C}$ given by $\gamma(t) = x(t) + iy(t)$. We will also require that γ is C^1, as we will require a well-defined tangent at every point of the curve.

GIven a function $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ and a contour $\Gamma \subset \Omega \subset \mathbb{C}$ parametrised by $\gamma : [a,b] \to \mathbb{C}$, the *contour integral* of f over Γ is given by:

$$\begin{aligned}\int_\Gamma f\,dz &:= \int_a^b f\big(\gamma(t)\big)\gamma'(t)\,dt \\ &= \int_a^b \Re\Big(f\big(\gamma(t)\big)\gamma'(t)\Big)\,dt + \int_a^b \Im\Big(f\big(\gamma(t)\big)\gamma'(t)\Big)\,dt\end{aligned}$$

If Γ is only piecewise C^1, then we define,

$$\int_\Gamma f\,dz := \sum_{i=1}^n \int_{\Gamma_i} f\,dz$$

where $(\Gamma_i)_{i=1}^n$ are the C^1 components of Γ.

Theorem 3.16. *Let $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ and $\Gamma \subset \Omega$ such that $f\big|_\Gamma$ is continuous, and parametrise Γ by $\gamma^+ : [a,b] \to \mathbb{C}$. Then,*

- *If γ^- represents the parametrisation of Γ in the opposite direction from γ^+, then,*

$$\int_{\gamma^-} f\,dz = -\int_{\gamma^+} f\,dz$$

If Γ has an attached notion of direction or orientation , we call it a directed curve or directed contour. In this case, we denote by $-\Gamma$ the same curve swept in the opposite direction, so we may reformulate the above result without reference to any particular parametrisation by:

$$\int_{-\Gamma} f\,dz = -\int_\Gamma f\,dz$$

- *If $\tilde{\gamma} : [\tilde{a},\tilde{b}] \to \mathbb{C}$ is another parametrisation of Γ that preserves orientation, then,*

$$\int_{\tilde{\gamma}} f\,dz = \int_\gamma f\,dz$$

This property is called reparametrisation invariance.

Given a function $f : \mathbb{C} \to \mathbb{C}$ and a curve parametrised by $\gamma : [a,b] \to \mathbb{C}$, we define,

$$\int_\gamma f\,d\bar{z} := \int_a^b f\big(\gamma(t)\big)\overline{\gamma'(t)}\,dt$$

Note that, unlike for functions $f : [a,b] \to \mathbb{C}$, in general, for contour integrals,

$$\overline{\int_\gamma f(z)\,dz} \neq \int_\gamma \overline{f(z)}\,dz$$

We instead have,

$$\overline{\int_\gamma f(z)\,dz} = \int_\gamma \overline{f(z)}\,d\bar{z}$$

Given a function $f : \mathbb{C} \to \mathbb{C}$ and a curve parametrised by $\gamma : [a,b] \to \mathbb{C}$, we define,

$$\int_\gamma |f||dz| := \int_a^b \Big| f\big(\gamma(t)\big)\Big| \big|\gamma'(t)\big|\,dt$$

Note that $\int_\gamma |f||dz| \geq 0$, so we have,

$$\left|\int_\gamma f\,dz\right| \leq \int_\gamma |f||dz|$$

If $f(z) = 1$, then we also have,

$$\int_\gamma |dz| = L(\gamma)$$

where $L(\gamma)$ is the length of γ.

Theorem 3.17. *Suppose that Ω is an open set, and $F : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ is holomorphic, such that $f(z) := \frac{dF}{dz}$ is continous. Let $\gamma : [a,b] \to \Omega$ be a C^1 curve. Then,*

$$\int_\gamma f\,dz = F\big(\gamma(b)\big) - F\big(\gamma(a)\big)$$

A set $\Omega \subset \mathbb{C}$ is *connected* if it cannot be expressed as the union of non-empty open sets Ω_1 and Ω_2 such that $\Omega_1 \cap \Omega_2 = \emptyset$.

An open connected set $\Omega \subset \mathbb{C}$ is *simply connected* if every closed curve in Ω can be continuously deformed to a point (more precisely, every closed curve is homotopic to a constant function). An example of a set that is not simply connected is an annulus (a 2D torus; a ring): any closed curve that encircles the hole cannot be continuous deformed into a point as it must always encircle the hole.

Theorem 3.18 (Cauchy)**.** *Let $f : \Omega \to \mathbb{C}$ be holomorphic, with Ω open and simply connected, and let $\gamma \subset \Omega$ be a C^1 closed curve. Then,*

$$\int_\gamma f(z)\,dz = 0$$

A parametrisation of a simple closed curve is *positively oriented* if, when following the direction of parametrisation, the interior is to our left, and is *negatively oriented* otherwise. For example, the counterclockwise parametrisation of the unit circle given by $\gamma(t) = (\cos(t), \sin(t))$ is positively oriented.

However, take an annulus, for example. This region has two boundary curves; an *exterior* and *interior* boundary. The exterior boundary is positively oriented if it has a counterclockwise parametrisation, but the interior boundary is positively oriented if it has a clockwise parametrisation.

Theorem 3.19 (Deformation of Contours)**.** *Let $\Omega \subset \mathbb{C}$ be a region bounded by two simple curves, γ_1 exterior and γ_2 interior, both oriented positively, and let f be a function holomorphic over $\Omega \cup \gamma_1 \cup \gamma_2$. Then,*

$$\int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz = 0$$

If we denote by γ_2^- the counterclockwise parametrisation of γ_2, then,

$$\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2^-} f(z)\, dz$$

That is, the integral is the same along both curves when both are parametrised in the same direction.

Given a simple closed C^1 curve γ, we denote by $I(\gamma)$ the region interior to γ, and by $O(\gamma)$ the region exterior to γ.

Theorem 3.20. *Let $\gamma : [a,b] \to \mathbb{C}$ be a positively oriented simple closed C^1 curve, and suppose f is a function holomorphic over $\gamma \cup I(\gamma)$. Then, for all $z \in I(\gamma)$,*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z}\, dw$$

This theorem says that we can recover the value of f at any point z by integrating along a closed curve around that point, given some restrictions on the curve.

Theorem 3.21. *Let $\gamma : [a,b] \to \mathbb{C}$ be a positively oriented simple closed C^1 curve, and suppose f is a function holomorphic over $\gamma \cup I(\gamma)$. Then, for all $z \in I(\gamma)$, f is smooth (infinitely differentiable), and the nth derivative is given by*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w - z)^{n+1}}\, dw$$

Theorem 3.22 (Taylor Series Expansion)**.** *Let f be holomorphic on $\mathbb{B}_r(a)$ for $a \in \mathbb{C}$, $r > 0$. Then, there exist unique constants c_n, $n \in \mathbb{N}$ such that, fo all $z \in \mathbb{B}_r(a)$*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$$

That is, a holomorphic function is analytic.

Moreover, the coefficients c_n are given by

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w - a)^{n+1}}\, dw \\ &= \frac{f^{(n)}(a)}{n!} \end{aligned}$$

where γ is any positively oriented parametrisation of a simple closed curve $\Gamma \subset \mathbb{B}_r(a)$ that is piecewise C^1 with $a \in I(\gamma)$.

Theorem 3.23 (Liouville)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be entire (analytic over $\mathbb{C}$) and bounded. Then, f is constant.*

Proof. Given two points x and y, consider the open balls $\mathbb{B}_r(x)$ and $\mathbb{B}_r(y)$, where $r > |x - y|$. For sufficiently large r, the two balls coincide except for an arbitrarily small proportion of their volume. Since f is bounded and entire functions are harmonic, by the mean value property, the averages of f over the two balls are arbitarily close so f takes the same value at x and y. Since x and y were arbitrary, f is constant. ∎

Theorem 3.24 (Fundamental Theorem of Algebra)**.** *Every non-constant polynomial $p \in C[x]$ has a root in $\mathbb{C}$ – that is, there exists $\alpha \in \mathbb{C}$ such that $p(\alpha) = 0$.*

Theorem 3.25. *Let Ω be open, and let $f_n : \Omega \to \mathbb{C}$ be a sequence of analytic functions. If f_n converges uniformly to f, then f is analytic.*

A function f defined on a subset of $\mathbb{C}$ is said to have a *pole* of order $m \in \mathbb{N}$ at $a \in \mathbb{C}$ if there is a neighbourhood U of a such that for any $z \in U$,

$$f(z) = \frac{c_{-m}}{(z-a)^m} + \frac{c_{m-1}}{(z-a)^{m-1}} + \cdots + \frac{c_{-2}}{(z-a)^2} + \frac{c_{-1}}{(z-a)} + \phi(z)$$

where ϕ is analytic in U, $(c_{-k})_{k=1}^m \subset \mathbb{C}$, and $c_{-m} \neq 0$. The coefficient c_{-1} is called the *residue* of f at a, also denoted $\operatorname{Res}\big(f(a)\big)$. This expansion is also called a *Laurent polynomial.*

A function that is holomorphic at all points of an open subset $\Omega \subset \mathbb{C}$ apart from some poles is said to be *meromorphic* on Ω.

Theorem 3.26 (Cauchy's Residue Theorem)**.** *Let $\gamma \subset \mathbb{C}$ be a simple closed positively oriented piecewise C^1 curve, and let f be meromorphic on $I(\gamma)$ with poles $(z_k)_{k=1}^n \subset I(\gamma)$. Then,*

$$\int_\gamma f = 2\pi i \sum_{k=1}^n \operatorname{Res}\big(f(z_k)\big)$$

Lemma 3.27. *Let f,$g : U \to \mathbb{C}$ be holomorphic on an open neighbourhood U of $a \in \mathbb{C}$, and suppose $g(a) = 0$, but $g'(a) \neq 0$. Then, provided $f(a) \neq 0$, the function $\frac{f}{g}$ has a pole of order 1 at a, and,*

$$\operatorname{Res}\left(\frac{f}{g}(a)\right) = \frac{f(a)}{g'(a)}$$

Example. Compute,

$$\int_{-\infty}^{\infty} \frac{1}{z^2+1}\, dz$$

We factorise the integrand into,

$$f(z) = \frac{1}{(z-i)(z+i)}$$

and we can see that f has a pole at $z = i$ and at $z = -i$, and is analytic elsewhere. We compute the residues of the poles:

$$\begin{aligned}
f(z) &= \frac{1/(z+i)}{(z-i)} \\
\operatorname{Res}[f(z)]_{z=i} &= \frac{1}{z+i} \\
&= \frac{1}{2i} \\
f(z) &= \frac{1/(z-i)}{(z+i)} \\
\operatorname{Res}[f(z)]_{z=-i} &= \frac{1}{z-i} \\
&= -\frac{1}{2i}
\end{aligned}$$

(We only need one of these, but both have been shown for purposes of illustration.)

Next, we create a positively oriented contour by following along the real line from $-R$ to R, then closing the contour with a semicircular arc in the complex plane, so we have,

$$\oint_\Gamma \frac{1}{z^2+1}\,dz = \int_{-R}^{R} \frac{1}{z^2+1}\,dz + \int_{\text{arc}} \frac{1}{z^2+1}\,dz$$

The resulting loop contour also encloses the pole at $z = +i$, so by the residue theorem, we have,

$$\begin{aligned}\oint_\Gamma \frac{1}{z^2+1}\,dz &= 2\pi i\,\text{Res}[f(z)]_{z=i}\\ &= 2\pi i\frac{1}{2i}\\ &= \pi\end{aligned}$$

$$\pi = \int_{-R}^{R} \frac{1}{z^2+1}\,dz + \int_{\text{arc}} \frac{1}{z^2+1}\,dz$$

We parametrise the arc as $Re^{i\theta}$ for $0 \leq \theta \leq \pi$,

$$\begin{aligned}\int_{\text{arc}} \frac{1}{z^2+1}\,dz &= \int_0^\pi \frac{1}{(Re^{i\theta})^2+1}\frac{dz}{d\theta}\,d\theta\\ &= \int_0^\pi \frac{1}{R^2e^{i2\theta}+1} iRe^{i\theta}\,d\theta\\ &= \int_0^\pi \frac{iRe^{i\theta}}{R^2e^{i2\theta}+1}\,d\theta\\ \left|\int_{\text{arc}} \frac{1}{z^2+1}\,dz\right| &\leq \int_0^\pi \left|\frac{iRe^{i\theta}}{R^2e^{i2\theta}+1}\right|\,d\theta\\ &= \int_0^\pi \frac{R}{|Re^{i2\theta}+1|}\,d\theta\\ &\leq \int_0^\pi \frac{R}{R^2-1}\,d\theta\\ &= \pi\frac{R}{R^2-1}\end{aligned}$$

so this integral vanishes as $R \to \infty$, leaving,

$$\int_{-\infty}^{\infty} \frac{1}{z^2+1}\,dz = \pi$$